Virtually biautomatic groups

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Abstract

We introduce *virtually biautomatic groups* (groups with finite index biautomatic subgroups) and generalize results of Gersten and Short [3] and Mosher [5] on centralizers, normalizers, and quotients to virtually biautomatic groups.

1 Introduction

A biautomatic group G, roughly, is a finitely-generated group such that the problem of whether two words $w_1, w_2 \in \mathcal{L}$ in the generators differ by left or right-multiplication by a given element is solvable by a finite-state automaton. Here, $\mathcal{L}$ is some fixed collection of words such that all elements of G are represented by some word in $\mathcal{L}$, and so that $\mathcal{L}$ is the set of accepted words by a finite-state automaton.

It is easy to see that, if a group G has a finite-index automatic subgroup H, then it is automatic. The corresponding result for biautomatic groups is unknown, and, by our Remark 2.3, even if it is true, one cannot always extend the biautomatic structure for H to one for G. Motivated by this, we define *virtually biautomatic groups* to be groups with a finite-index biautomatic subgroup, and investigate the properties of these perhaps more general groups.

In [3], Gersten and Short proved that centralizers of finite sets $S \subset G$ in biautomatic groups are themselves biautomatic. In [5], Mosher proved that, if G is biautomatic, then G/Z is biautomatic for any finitely-generated central subgroup $Z < G$. Here we generalize these results in the setting of virtually biautomatic groups. The main technical result is the following:

Theorem 3.4. *Let G be a virtually biautomatic group and $K \leq G$ a finitely generated subgroup such that $[K : Z(K)]$ is finite. Then $[N_G(K) : Z_G(K)]$ is finite.*

As a consequence, we deduce the following, which partially extends a result on centralizers from [3] to normalizers. Here, a subgroup $J < H$ is $\mathcal{L}$-rational if the subset of $\mathcal{L}$ of elements mapping to J is also accepted by a finite-state automaton.

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Theorem 3.6. *Let $(H, \mathcal{A}, \mathcal{L})$ be biautomatic. Let $K < H$ be any finitely generated subgroup with finite index center. Then the normalizer $N_H(K)$ is $\mathcal{L}$-rational, and thus biautomatic.*

Theorem 3.7. *If G is virtually biautomatic and $H \leq G$ is a finitely-generated normal subgroup with finite index center, then G/H is also virtually biautomatic.*

Furthermore, we use our results to give an elementary proof of one of the main theorems of Gersten and Short [3] (generalized to virtually biautomatic groups):

Theorem 3.5. *Polycyclic subgroups of a virtually biautomatic group are virtually abelian.*

We begin by recalling some definitions (following [3] and its references, such as [2]) to fix notation.

Definition 1.1. *A finite state automaton is a quintuple $(Q, \mathcal{A}, \delta, q_0, F)$ where Q is a finite set of states, $\mathcal{A}$ is a finite alphabet, $q_0 \in Q$ is the initial state, $F \subset Q$ is the set of final states, and $\delta : Q \times \mathcal{A} \to Q$ is the transition function.*

In some of the literature, δ is allowed to be multivalued, but we lose no generality by taking it to be a function. Given a finite alphabet $\mathcal{A}$ we let $\mathcal{A}^*$ denote the set of all words in the alphabet $\mathcal{A}$, and refer to subsets of $\mathcal{A}^*$ as *languages* in $\mathcal{A}$. The transition function δ can be extended to $Q \times \mathcal{A}^*$ in the obvious way.

Definition 1.2. *A word $w \in \mathcal{A}^*$ is accepted by the finite state automaton $(Q, \mathcal{A}, \delta, q_0, F)$ if $\delta(q_0, w) \in F$. A language $\mathcal{L} \subseteq \mathcal{A}^*$ is called regular if there is some finite state automaton having $\mathcal{L}$ as its set of accepted words.*

If $\mathcal{A} \subseteq G$ there is a natural map from words in $\mathcal{A}^*$ to elements of G, sending the word $a_1 a_2 \dots a_n$ to the group element $a_1 \cdot a_2 \cdots a_n$ and the empty word to the identity element of G. We call this the *evaluation map*, $\mu : \mathcal{A}^* \to G$.

Definition 1.3. *A rational structure for a group G is a pair $(\mathcal{A}, \mathcal{L})$ where $\mathcal{A} \subseteq G$ is a finite set and $\mathcal{L} \subseteq \mathcal{A}^*$ is a regular language such that $\mu(\mathcal{L}) = G$.*

Definition 1.4. *An automatic structure for a group G is a rational structure $(\mathcal{A}, \mathcal{L})$ such that $\mathcal{L}_g := \{(u, v) \in \mathcal{L}^\$ | \mu(u) = \mu(v)g\}$ is regular for each $g \in \mathcal{A} \cup \{1\}$. A group is automatic if it has an automatic structure.*

Above, $\mathcal{L}^\$:= (\mathcal{L} \times \mathcal{L}\{\$\}^*) \cup (\mathcal{L}\{\$\}^* \times \mathcal{L})$ is the padded product language (cf. [3], p. 135), and $\mu(\$)$ is the identity.

Definition 1.5. *A biautomatic structure for a group G is an automatic structure $(\mathcal{A}, \mathcal{L})$ such that $\mathcal{L}'_g = \{(u, v) \in \mathcal{L}^\$ | \mu(u) = g\mu(v)\}$ is also regular for each $g \in G$.*

Definition 1.6. *An $\mathcal{L}$-rational subgroup of a biautomatic group G with automatic structure $(\mathcal{A}, \mathcal{L})$ is a subgroup $H < G$ such that $\mu^{-1}(H) \cap \mathcal{L}$ is regular.*

Finally we recall some results on biautomatic groups, which we will use throughout.

Proposition 1.7. *[3] $\mathcal{L}$-rational subgroups of (bi)automatic groups are (bi)automatic.*

Proposition 1.8. *[3] The centralizer of a finite subset of a biautomatic group is $\mathcal{L}$-rational for any biautomatic structure $(\mathcal{A}, \mathcal{L})$.*

2 Virtually biautomatic groups

Recall that a group G is said to be *virtually biautomatic* if it has a finite-index biautomatic subgroup.

Definition 2.1. *Let G be virtually biautomatic. A virtually biautomatic structure for G is a triple $(H, \mathcal{A}, \mathcal{L})$ where $H \leq G$ is a finite index subgroup with biautomatic structure $(\mathcal{A}, \mathcal{L})$. A subgroup $K \leq G$ is a virtually $\mathcal{L}$-rational subgroup of G if $K \cap H$ is an $\mathcal{L}$-rational subgroup of H.*

Any virtually biautomatic group is automatic. In fact, we can extend the biautomatic structure for H to an automatic structure for G. Let $(H, \mathcal{A}, \mathcal{L})$ be a virtually biautomatic structure for G. Choose a set $\{g_1, g_2, \dots, g_k\}$ of representatives for the nontrivial right cosets of H in G. Set $\mathcal{B} = \mathcal{A} \cup \{g_1, g_2, \dots, g_k\}$ and $\mathcal{L}' = \mathcal{L} \cup \{\ell g_i \mid \ell \in \mathcal{L}\}$. Then $(\mathcal{B}, \mathcal{L}')$ is an automatic structure for G. Note that $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{L} = \mathcal{L}' \cap \mu^{-1}(H)$ where $\mu : \mathcal{L}' \to G$ is the evaluation map.

Proposition 2.2. *Let $(H, \mathcal{A}, \mathcal{L})$ be a virtually biautomatic structure for G and let $(\mathcal{B}, \mathcal{L}')$ be an automatic structure for G constructed as above with $\mathcal{L} = \mathcal{L}' \cap \mu^{-1}(H)$.*

(i) $K \leq G$ is virtually $\mathcal{L}$-rational if and only if K is $\mathcal{L}'$-rational.

(ii) Let $J < K < G$ where $[K : J] < \infty$. Then J is $\mathcal{L}'$-rational if and only if K is $\mathcal{L}'$-rational.

(iii) If K is virtually $\mathcal{L}$-rational then K is virtually biautomatic.

(iv) If $S \subset H \leq G$ where S is finite then $Z_G(S)$ is virtually $\mathcal{L}$-rational.

(v) The center of a virtually biautomatic group is biautomatic.

Proof. (i) If K is virtually $\mathcal{L}$-rational then, by definition, $K \cap H$ is an $\mathcal{L}$-rational subgroup of H. Thus $\mu^{-1}(K \cap H) \cap \mathcal{L} = \mu^{-1}(K) \cap \mu^{-1}(H) \cap \mathcal{L}$ is regular. Since $\mathcal{L} = \mathcal{L}' \cap \mu^{-1}(H)$ we have $\mu^{-1}(K) \cap \mu^{-1}(H) \cap \mathcal{L}' = \mu^{-1}(K \cap H) \cap \mathcal{L}'$ is regular, hence $K \cap H$ is $\mathcal{L}'$-rational. We now defer to part (ii) to deduce that K is $\mathcal{L}'$-rational, since $[K : K \cap H] \leq [G : H] < \infty$.

Conversely, assume that K is an $\mathcal{L}'$-rational subgroup of G. Then $\mu^{-1}(K) \cap \mathcal{L}'$ is regular. From the virtually biautomatic structure on G we have that $\mathcal{L} = \mu^{-1}(H)$ is regular. Hence the intersection $\mu^{-1}(K) \cap \mathcal{L}' \cap \mu^{-1}(H) \cap \mu^{-1}(H) = \mu^{-1}(H \cap K) \cap \mathcal{L}$ is regular, and K is virtually $\mathcal{L}$-rational.

(ii) For this part, we do not need the virtually biautomatic structure, only the automatic structure on G. We first show the "only if" direction. Suppose that J is $\mathcal{L}'$-rational. Then, $\mathcal{L}'' := \mu^{-1}(J) \cap \mathcal{L}'$ is regular. Let $[K : J] = m$ and let $\{k_1, \dots, k_{m-1}\}$ be a set of representatives for the nontrivial right cosets of J in K. Now $\mathcal{L}_i := \mu^{-1}(Jk_i)$ is the projection to the second component of the regular language

$$\{(u, v) \mid u, v \in \mathcal{L}', \mu(v) = \mu(uk_i)\} \cap (\mathcal{L}'' \times \mathcal{L}').$$

Hence $\cup_i \mathcal{L}_i \cup \mathcal{L}'' = \mu^{-1}(K) \cap \mathcal{L}'$ is regular, so K is $\mathcal{L}'$-rational.

We next show the "if" direction. Using the "only if", and the fact that K is finitely-generated, it suffices to assume that J is normal in K (by intersecting J with its conjugates by generators of K). Now, we follow a construction from the proof of Proposition 2.3 in [3], to argue that J is $\mathcal{L}'$-rational. Since K is $\mathcal{L}'$-rational, by Theorem 2.2 of loc. cit., there exists $k \geq 0$ so that, for all words $a_1a_2\cdots a_m \in \mathcal{L}'$, there exists $g_m \in G$ with $|g_m|_{\mathcal{A}} \le k$ such that $a_1a_2\cdots a_m g_m \in K$. Then, similarly defining g_i for $i < m$, we have that

$$(a_1g_1)(g_1^{-1}a_2g_2)\cdots(g_{m-1}^{-1}a_mg_m) \in K, \tag{1}$$

so that elements of the form $g_{i-1}^{-1}a_ig_i^{-1} \in K$ (for $|g_{i-1}|_{\mathcal{A}}, |g_i|_{\mathcal{A}} \le k, a_i \in \mathcal{A}$) form a finite generating set $\mathcal{B}$ for K. Furthermore, by Theorem 3.1 of loc. cit., the rewritten elements of $\mathcal{L}'$ form a regular language $\mathcal{L}''$ for the alphabet $\mathcal{B}$, which is an automatic structure for K. It is clear that J is a rational subgroup of K with respect to this automatic structure (since J is normal of finite index, i.e., K/J is finite, so we need only consider the images of $\mathcal{B}$ in K/J). Now, from any automaton that accepts the regular language $\mathcal{L}'' \cap \mu^{-1}(J)$, one may obtain one that accepts the language $\mathcal{L}' \cap \mu^{-1}(J)$ (e.g., the automaton given by Lemma 3.2 of loc. cit. does exactly this). Thus, J is $\mathcal{L}'$-rational.

(iii) and (iv) are immediate consequences of the results in [3].

(v) Note that H biautomatic implies $Z(H)$ biautomatic, hence finitely generated, by [3]. Now $Z(H) = Z_G(H) \cap H$, and $[Z_G(H) : Z(H)] = [HZ_G(H) : H] \le [G : H] < \infty$, so $Z_G(H)$ is finitely generated as well. Now $Z_G(H) \cap Z(H)$ is a subgroup of the finitely generated abelian group $Z(H)$, and hence is finitely generated abelian itself. Note that $[Z(G) : Z(G) \cap Z(H)] \le [Z_G(H) : Z(H)] < \infty$ so $Z(G)$ is finitely generated abelian as well, and hence biautomatic. □

Remark 2.3. It is not always true that the centers of virtually biautomatic groups are virtually rational. Indeed, consider $H = \mathbb{Z}^2$ and $G = Z_2 \ltimes_\phi H$ where $\phi(a,b) = (a, 2a-b)$. Consider the biautomatic structure $H = (\{a, a^{-1}, b, b^{-1}\}, \{a^ib^j, a^i, b^j, \epsilon | i, j \in \mathbb{Z}\setminus\{0\}\})$ where ϵ is the empty word. Then $Z(G) = \{(0, a, a) \mid a \in \mathbb{Z}\} \subset H$, which is clearly biautomatic but not $\mathcal{L}$-rational. Thus even if it turns out that all virtually biautomatic groups are biautomatic (which is still unknown), not all virtually biautomatic structures can be extended to biautomatic ones. Note this also gives an example of a biautomatic group $H < G$ where the biautomatic structure for H cannot extend to one for G.

We generalize the rationality of centralizers (from [3]) to stabilizers:

Proposition 2.4. *Let G have virtually biautomatic structure $(H, \mathcal{A}, \mathcal{L})$. Let $\rho : G \to \mathrm{Aut}(G)$ be the action by conjugation, i.e. $\rho(g)(h) = g^{-1}hg$. Let $S \subset H$ be any finite subset of H. Then the stabilizer $\mathrm{Stab}_G(S)$ under ρ is virtually $\mathcal{L}$-rational. Moreover if $\rho : \mathrm{Stab}_G(S) \to \mathrm{Aut}(S)$ is the map induced by ρ then, for any subgroup $K < \mathrm{Aut}(S)$, the inverse image $\rho^{-1}(K)$ is virtually $\mathcal{L}$-rational.*

Proof. Each $g \in \mathrm{Stab}_G(S)$ induces a permutation on S, and if g and g' induce the same permutation we have $g^{-1}sg = g'^{-1}sg'$, from which it follows that $g'g^{-1} \in Z_G(S)$. If $|S| = n$ there can be at most $n!$ different permutations of S, so we have that $[\mathrm{Stab}_G(S) : Z_G(S)] \le n!$, whence $[\mathrm{Stab}_G(S) \cap H : Z_G(S) \cap H] \le n!$. Now $Z_G(S)$ is virtually $\mathcal{L}$-rational by Proposition 2.2 (iv) and hence by Proposition 2.2 (i) and (ii) so is $\mathrm{Stab}_G(S)$.

Now let K be any (necessarily finite) subgroup of $\mathrm{Aut}(S)$. Note that $\rho^{-1}(1_{\mathrm{Aut}(S)}) = Z_G(S)$, hence $[\rho^{-1}(K) : Z_G(S)]$ is finite. The result now follows again by Proposition 2.2 (i) and (ii). □

3 The main theorem and consequences

We first recall some results from [3].

Definition 3.1. *[3] Let G be generated by a finite set $\mathcal{A}$. For any $g \in G$ define $\tau_{G,\mathcal{A}}(g)$, the translation number of g, to be $\tau_{G,\mathcal{A}}(g) = \lim_{n\to\infty} \frac{||g^n||}{n}$, where $||g^n||$ is the length of the shortest word representing the group element g^n.*

Definition 3.2. *[3] Let S be any set. For any two functions $f, g : S \to \mathbb{R}$ we say that $f \sim g$ if there exist positive constants λ, ϵ such that $\frac{1}{\lambda} f(s) - \epsilon \leq g(s) \leq \lambda f(s) + \epsilon$ for all $s \in S$. It is easy to check that this is an equivalence relation.*

Lemma 3.3. *[3]*

(i) The translation number is always well defined (the limit exists).

(ii) Let G be automatic with structure $(\mathcal{A}, \mathcal{L})$ such that $\mathcal{L}$ is a set of unique representatives of elements of G.[1] Let $H \leq G$ be an $\mathcal{L}$-rational subgroup with finite generating set $\mathcal{B}$. Then, restricted to H, $\tau_{G,\mathcal{A}} \sim \tau_{H,\mathcal{B}}$ (see the proof of Proposition 6.5 in [3]).

(iii) If A is a finitely generated free abelian group with basis $\mathcal{A} = \{a_1, \ldots, a_m\}$, then $\tau_{A,\mathcal{A}}(x) = ||x||_{\mathcal{A}} = ||x||_1$ the l_1 norm with respect to the basis $\mathcal{A}$ (namely $||(x_1, \ldots, x_m)|| = |x_1| + \ldots + |x_m|$, where the coordinates are in terms of the basis $\mathcal{A}$).

(iv) The translation number is invariant under conjugation, i.e. $\tau_{G,\mathcal{A}}(x) = \tau_{G,\mathcal{A}}(gxg^{-1})$ for all $x, g \in G$.

Now we prove Theorem 3.4:

Theorem 3.4. *Let G be a virtually biautomatic group and $K \leq G$ a finitely generated subgroup such that $[K : Z(K)]$ is finite. Then $[N_G(K) : Z_G(K)]$ is finite.*

Proof. Equivalently, we show that the image of the natural map $N_G(K) \to \mathrm{Aut}(K)$ given by conjugation is finite. We claim that $Z(K)$ is finitely-generated, and that it suffices to show that the image of the natural map $N_G(K) \to \mathrm{Aut}(mZ(K))$ is finite, for any integer $m \geq 1$.

First, for any finitely-generated group G with a finite-index normal subgroup H, H must be finitely generated (in fact, it must be generated by those words of length $\leq [G : H]$ in the generators of G, whose product is an element of H).

Next, under the same hypotheses, the set of automorphisms of G which restrict to the identity on H and the quotient G/H are determined by the images of representatives of each element of G/H (landing in the same coset). If, furthermore, H is central, then the image of any element $g \in G$ must be of the form $g \cdot h$ where $h \in H$ is a torsion element satisfying $h^r = 1$, where $r \geq 1$ is such that $g^r \in H$. So, in this case, the size of the set of automorphisms of G which restrict to the identity on H is at most $([G : H]!) \cdot |\operatorname{torsion}(H)|^{[G:H]}$, which is finite since H is finitely-generated abelian, and hence $|\operatorname{torsion}(H)| < \infty$.

Going back to our situation, it suffices to show that the image $N_G(K) \to \mathrm{Aut}(m \cdot Z(K))$ is finite (by setting $H := m \cdot Z(K)$ in the above paragraph; note that $[K : m \cdot Z(K)] < \infty$ and $m \cdot Z(K)$ is fixed by all automorphisms of K). For this, we use Lemma 3.3. Suppose that $(H, \mathcal{A}, \mathcal{L})$ is a virtually biautomatic structure for G; we assume that $\mathcal{L}$ is a set of unique representatives of G, as is possible by p. 135 of [3]. We may take $m \geq 1$ such that $m \cdot Z(K) < H$, since $[Z(K) : Z(K) \cap H] < \infty$.

[1] Such languages always exist for G (bi)automatic (see p. 135 of [3]), and the property is preserved under extending from $H < G$ of finite index to G, as in §2.

Let $J := Z(Z_H(K \cap H))$. We know that J is $\mathcal{L}$-rational, and is an abelian group containing $m \cdot Z(K)$. Now, pick generators $\mathcal{B}$ of J. By Lemma 3.3.(ii), $\tau_{G,\mathcal{A}} \sim \tau_{J,\mathcal{B}}$ when restricted to J. If we assume that $\mathcal{B}$ consists of generators for the torsion of J together with a basis for a free complement in J of torsion(J), then it is evident that there are only finitely many elements with $\tau_{J,\mathcal{B}}(x) \leq M$ for any fixed positive integer M; hence, also finitely many $x \in J$ with $\tau_{G,\mathcal{A}}(x) \leq M$. Thus, since translation number is fixed under conjugation, there can only be finitely many images of finitely many generators of $m \cdot Z(K) < J$ under conjugation by elements of $N_G(K)$. Hence, the image of $N_G(K) \to \mathrm{Aut}(m \cdot Z(K))$ is finite. □

This yields a simple proof of a theorem from [3] (generalized to virtually biautomatic groups):

Theorem 3.5. *Polycyclic subgroups of a virtually biautomatic group are virtually abelian.*

Proof. It suffices to show inductively that, for $B \lhd C < G$ where G is virtually biautomatic, B is finitely generated virtually abelian and C is an extension of B by $\mathbb{Z}$, then C must also be virtually abelian. Let $A < B$ be abelian and $[B : A] < \infty$. It suffices to suppose that A is normal in C. Then $[C : Z_C(A)] < \infty$ by Theorem 3.4. So there is an element $c \in Z_C(A) \setminus B$. Clearly, the group $\langle c, A\rangle$ is abelian and has finite index in C, as desired. □

Theorem 3.6. *Let $(H, \mathcal{A}, \mathcal{L})$ be biautomatic. Let $K < H$ be any finitely generated subgroup with finite index center. Then the normalizer $N_H(K)$ is $\mathcal{L}$-rational and thus biautomatic.*

Proof. By [3], $Z_H(K)$ is $\mathcal{L}$-rational. Theorem 3.4 implies that $[N_G(K) : Z_G(K)]$ is finite. Now, Proposition 2.2 (ii) establishes that $N_G(K)$ is $\mathcal{L}$-rational. □

Finally we have a result on quotients. Recall that Mosher [5] proved that given a biautomatic group G and a subgroup of the center $H \leq Z(G)$ then G/H is biautomatic.

Theorem 3.7. *If G is virtually biautomatic and $K \leq G$ is a finitely generated normal subgroup with finite-index center then G/K is also virtually biautomatic.*

Proof. By Theorem 3.4 we have $[G : Z_G(K)] < \infty$ since $N_G(K) = G$. Suppose $H < G$ is a finite-index biautomatic subgroup. Then, $J := Z_H(K \cap H) < H$ is biautomatic and has finite index in G. Since $(K \cap H) \cap J = J \cap K$ is central in J, by [5], $J/(J \cap K)$ is biautomatic. But, $[G/K : J/(J \cap K)]$ is finite, so G/K is virtually biautomatic. □

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